Quantum Hydrodynamics of Fractional Hall Effect: Quantum Kirchhoff Equations

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We argue that flows of the quantum electronic liquid in the Fractional Quantum Hall state are comprehensively described by the hydrodynamics of vortices in the quantum incompressible rotating liquid. We obtain the quantum hydrodynamics of vortex flow by quantizing Kirchhoff equations for vortex dynamics. We demonstrate that quantized Kirchhoff equations capture all major features of FQH states including subtle effects of Lorentz shear force, magneto-roton spectrum, Hall current in a non-uniform electromagnetic field, thus providing a powerful framework to study FQHE and superfluids.

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1. Introduction. In Fractional Quantum Hall (FQH) regime electrons form a peculiar quantum liquid. Characteristic features of the liquid are: flows are incompressible [1], almost dissipation-free [2, 3], the Hall conductance is quantized [2], vortices are elementary excitations, vortices carry fractionally quantized negative electronic charges [1], they are separated from the ground state by a gap [3, 4]. More subtle features are the universal anharmonic term of the structure factor and magneto-roton minimum in excitation spectrum [4] (see (24)), quantized double layers of a density and shear at boundaries and vortices [5], the Lorentz shear force [6].

Such liquid does not posses linear gapless waves except edge modes propagating along the boundary. Only available bulk flow is a non-linear flow of vorticity. Since in FQH regime vorticity is linked to the electronic charge, flows of vorticity are related to a charge flow [25].

A natural approach to study flows of FQH states is hydrodynamics. Not all quantum liquids are subject of quantum hydrodynamics. Quantum hydrodynamics is based on a fundamentally restrictive assumption that all quantum states are fully characterized by the density and velocity. Apart from of superfluids (and superconductors) and Luttinger liquids, FQH liquid is, yet, another important case.

Hydrodynamics of quantum fluids goes back to Landau [7] and Feynman [8], but remains in its infancy. A quest for the hydrodynamics of FQH liquid has been originated by a seminal paper [4], further discussed in [9–12], it is a focus of a renewed interest.

As always in hydrodynamics only a few basic principals, symmetries, and a few phenomenological parameters are sufficient to formulate fundamental equations. To this aim an underlying microscopic Hamiltonian describing emergent FQH states under a strong Coulomb interaction is in fact not necessary.

In the letter we attempt to formulate these principals and to develop hydrodynamics of FQH bulk states in a close analog of Feynman theory of superfluid helium [8] and magneto-roton theory of collective excitations by [4].

We argue that flows of FQH liquid are equivalent to flows of vortices in the quantum incompressible rotating Euler liquid. Based on this conjecture we obtain the major features od FQHE including subtle effects of Lorentz shear force [6]and magneto-roton spectrum [4] missed by the previous hydrodynamics approaches [9–12].

Hydrodynamics of vortex flows (classical and quantum alike) is an interesting subject by its own. Apart from FQHE it is also relevant to the theory of superfluids and classical hydrodynamics.

Quantization of hydrodynamics is a subtle matter. It is best achieved through quantization of Kirchhoff equations [13]. Classical Kirchhoff equations describe a motion of vortices in a 2D incompressible isentropic fluid. We show that, quite remarkable, quantum Kirchhoff equations capture all known features of FQH liquid. They can be used as a platform for studying FQH Effect.

In this paper we consider only the simplest Laughlin's cases where fraction ν is an inverse of an odd integer, say 1/3. In these cases electronic liquids do not posses additional symmetries. Extension to other FQH states, not discussed here, is possible and interesting.

We start by reminding classical Kirchhoff equations for rotating incompressible Euler flows (see e.g.,[14]).

2. Classical Kirchhoff equations. 2D incompressible isentropic flow (that is a flow where gradient of density are orthogonal to gradients of pressure, or zero) is fully characterized by its vorticity. Vorticity obeys a single (pseudo) scalar equation, which in the case of inviscid fluid has a simple geometrical meaning: the material derivative of the vorticity vanishes

$$D_t\varpi \equiv \left(\frac{\partial}{\partial t} + u\cdot\nabla\right)\varpi = 0. \qquad (1)$$

In other words vorticity $\varpi = \nabla\times u$ is transported along divergent-free velocity field u

$$\nabla\cdot u = 0. \qquad (2)$$

Helmholtz, and later Kirchhoff realized that there is a class of solutions of the vorticity equation (1) which consists of a finite number of point-like vortices moving by Magnus forces. In this case the complex velocity

$\mathrm{u} = u_x - iu_y$ is a meromorphic function

$$\mathrm{u}(z,t) = -\mathrm{i}\Omega\bar{z} + \mathrm{i}\sum_{i=1}^{N}\frac{\Gamma_i}{z - z_i(t)}, \tag{3}$$

where Ω is an angular velocity if the fluid is rotated, N is a number of vortices, Γ_i and $z_i(t)$ are circulations and positions of vortices. We denote $z = x + \mathrm{i}y, \partial = \frac{1}{2}(\nabla_x - \mathrm{i}\nabla_y)$ and use the roman script for complex vectors $\mathrm{a} = a_x - ia_y$.

A substitution this "pole Ansatz" into the Euler equation (1) yields that the number of vortices N and the circulations Γ_i do not change in time, but moving position of vortices $z_i(t)$ obey the Kirchhoff equations.

If the rotation of fluid is very strong, vortices prefer to be of the same sign opposite to the direction of the rotation. Bearing in mind the quantum case we assume that vortices have the same (minimal) circulation $\Gamma_i = \Gamma$. Then Kirchhoff equations obtained by localizing the Euler equation to vortex cores at $z = z_i$ expresses velocities v_i of vortex cores through their positions

$$\mathrm{v}_i \equiv \dot{\bar{z}}_i = -\mathrm{i}\Omega\bar{z}_i + \mathrm{i}\sum_{i\neq j}^{N}\frac{\Gamma}{z_i(t) - z_j(t)}. \tag{4}$$

The Kirchhoff equations replace the non-linear PDE (1) by the dynamical system, reflecting integrability of the Euler flow. They can be used for different aims. Equations describe chaotic motions of a finite number of vortices if $N > 3$. Alternatively they can be used to approximate virtually any flow if $N \to \infty$.

3. Chiral flow. The flow relevant for FQHE is the *chiral flow,* where a large number of background vortices uniformly distributed with the mean density $\bar{\rho} = \Omega/(\pi\Gamma)$ largely compensates rotation. This is a very special flow in fluid mechanics. There we distinguish two types of motion: fast motion of the fluid around vortex cores, and a slow motion of vortices. In this respect vortices themselves must be considered as a (secondary) fluid. At the ground state of chiral flow vortices do not move, but the fluid does. Vortices carry an inertia m_* – a phenomenological parameter determined by the equation of state. It is not directly related to the inertia of fluid.

In this paper we propose to model FQHE by a quantized slow flow of vortex fluid.

4. Quantization of Kirchhoff equations. Quantization of the Kirchhoff equations consists of three steps: canonical quantization, a choice of the representation and the inner product.

It is convenient to use a circulation $q = m_*\Gamma$ of momenta of vortices m_*v_i. The Poisson brackets followed from canonical Hamiltonian structure of the hydrodynamics. They are equal to the volume per particle per circulation $\{x_i,\, y_j\}_{P.B.} = 1/(2\pi\bar{\rho}q)$. We replace them by commutators

$$\{\bar{z}_i,\, z_j\}_{P.B.} \to [\bar{z}_i,\, z_j] = 2\ell^2\delta_{ij}. \tag{5}$$

Here $2\ell^2 = \nu/(\pi\bar{\rho})$ has a dimension of area. Dimensionless number $\nu = \hbar/q$ is a semiclassical parameter.

The next step is a choice of states. We assume that states are holomorphic polynomials of z_i. Then operators $\bar{z}_i$ are canonical momenta

$$\bar{z}_i = 2\ell^2\partial_{z_i}. \tag{6}$$

The third step is to specify the inner product. We impose the *chiral condition*: operators $\bar{z}_i$ and z_i are assumed to be Hermitian conjugated

$$\text{chiral condition}: \quad \bar{z}_i = z_i^\dagger. \tag{7}$$

This condition combined with representation (6) identifies a set of states with the Bargmann space [16]: the Hilbert space of analytic polynomials $\psi(z_1,\ldots,z_N)$ with the inner product $(\exp(-z\bar{z})\bar{\partial}\exp(z\bar{z}) = z$ insures (6,7))

$$\langle\psi'|\psi\rangle = \int d\mu\overline{\psi'}\psi, \quad d\mu = \prod_i e^{-\frac{|z_i|^2}{2\ell^2}}d^2z_i. \tag{8}$$

Consequently $\bar{z}_i^\dagger = -2\ell^2\bar{\partial}_{z_i} + z_i$. The normal ordering in the Bargmann space assumes that holomorphic operators stays to the left to anti-holomorphic operators.

Eqs. (4,6) help to write velocity operators of vortices

$$\Gamma^{-1}\mathrm{v}_i = -\mathrm{i}2\nu\partial_{z_i} + \mathrm{i}\sum_{i\neq j}\frac{1}{z_i - z_j}, \tag{9}$$

Eqs.(4-9) are quantum chiral Kirchhoff equations. They are readily generalized to a sphere, or a torus.

5. Quantum Chiral Kirchhoff Equations and FQHE. We identify quantum chiral Kirchhoff equations with FQHE.

First we comment that Bargmann space of analytic polynomials is another way to say that all states belong to the lowest Landau level [16] (and also [4, 17]). Then we assume that a state where vortices are at rest is the ground state of the system ψ_0. It nulls all velocity operators $\mathrm{v}_i\,\mathrm{Arg}\,[\psi_0] = 0$ (velocity operator acts on the phase of the w.f.). A common solution of the set of 1st order PDEs is the holomorphic part of the Laughlin w.f.

$$\psi_0 = \prod_{i>j}(z_i - z_j)^\beta, \qquad \hbar\beta = q. \tag{10}$$

The w.f. becomes single valued if β is integer, antisymmetric if β is an odd-integer.

In this interpretations vortices are identified with "particles" entered into the Laughlin function. Hence a quasihole $\psi_h = \prod_i(z_0 - z_i)\psi_0$ [1] is a hole in the uniform background of vortices - an anti-vortex. Thus we assign the electronic charge to vortices and identify angular velocity with the cyclotron frequency of vortices $\Omega = eB/(m_*c)$. Then the entries of the Kirchhoff equations are the magnetic length $\ell = \sqrt{\hbar c/eB}$ and the filling fraction $\nu = \bar{\rho}hc/eB = \hbar/q$ $(\Gamma = \Omega/(\pi\bar{\rho})$.

The phenomenological parameter m_* is the inertia of the vortex. It is naturally to assume that the energy

associated with the inertia $\Delta_\nu = \hbar\Omega = \hbar^2/(\ell^2 m_*)$ is of the same order as the energy of a quasi-hole at rest, or equivalently, is a gap in the excitation spectrum. The latter is of the order of Coulomb energy e^2/ℓ. It is known experimentally $\Delta_\nu \sim 10K$ [3]. The very existence of the FQH state requires that this energy scale must be less than the cyclotron frequency $\hbar\omega_c \sim 25meV \gg \hbar\Omega$, or that m_* exceeds the bare electronic mass $m_* \gg m_e$. An assumption that the energy of the quasi-hole at rest is the gap and is related to the inertia of the flow is the major physical input led to the hydrodynamics of FQHE.

A noticeable feature of the identification is that slowly varying external fields (potential well, gradients of temperature etc.) are coupled to slow moving vortices. For example a potential well $U(r)$ adds to the energy $\sum_i U(r_i)$, where r_i are coordinates of vortices, not fluid particles. It yields the Lorentz force $-i[U, \bar z_i] = i2\ell^2\partial_{z_i}U$ into the r.h.s. of (9). In the flow which keeps the center mass at the origin, Kirchhoff equations yield $\sum_i \mathrm{v}_i = \mathrm{i}(\pi q\bar\rho)^{-1}\sum_i \partial_{z_i}U$. This implies the fractionally quantized Hall conductance $\sigma_{xy} = e^2/(2\pi q) = \nu(e^2/h)$.

Finally we emphasize that the velocity (9) is different than velocity of individual electrons and coordinates of their guiding centers.

In the rest of the paper we show that Kirchhoff equations contain other, more subtle properties of FQHE. To this end we must develop the hydrodynamics of quantum vortex flow. To the best of our knowledge this has not been done even for the classical fluids.

6. Velocity field of the flow of vortices. Eulerian hydrodynamics of vortex computes the flux of vortices

$$P = \frac{m_*}{2}\sum_i\{\delta(r-r_i), v_i\} = \frac{m_*}{2}\{\rho, v\} = m_*\sqrt{\rho}\, v\,\sqrt{\rho}, \quad (11)$$

where r_i, v_i are coordinates and velocities of vortices, $\{,\}$ is the anti-commutator and $\rho(r) = \sum_i \delta(r - r_i)$ is the density of vortices. This formula also defines the velocity of the vortex flow $v(r)$.

By construction operators P and $\mathrm{P}^\dagger$ are ladder operators. They annihilate the ground state $\mathrm{P}\psi_0 = \bar\psi_0\mathrm{P}^\dagger = 0$.

In hydrodynamics the mass density of the fluid and mass flux of the flow are independent fields obeying the continuity and the Euler equations. Contrary, the velocity of the vortex flow $v(r)$ (as well as the velocity of the fluid $u(r)$) are expressed through the density of vortices $\rho(r)$. We must determine these relations. Before we proceed the comment is in order.

In 2D Euler liquid the mass density of the fluid and mass density of vortices, both, obey the continuity equations. Hence an initially imposed local condition, $n(r) = m_*\rho(r)$ is compatible with the dynamics at all time. This condition reduces the set of states being essentially equivalent to the chiral constraint (7). Thus the vortex flow can be formally considered as a special flow of 2D Euler liquid with a constituency relation between the flux and the density of the liquid [26]. From this point of view $J = \frac{1}{2}\{n, u\} = \frac{m_*}{2}\{\rho, u\}$ is the mass flux of the special flow. It includes fast fluid motion around vortices and a slow motion of vortices. This is a useful but an auxiliary object. Instead we are after the flux of the vortex flow (11), which consists of only slow motion of vortices.

The flux of the vortex flow subtly differs to the flux of the fluid. We compute them now.

The velocity of the fluid is easy obtain. It is given by (3), where we represent the operator $\bar z$ by (6), use $\rho\nabla\frac{\delta}{\delta\rho} = \sum_i \delta(r-r_i)\nabla_{r_i}$, and pass to the continuum limit

$$m_*\mathrm{u} = 2\partial\pi_\rho + \mathrm{i}q\int\frac{\rho d^2\xi}{z-\xi}, \quad \pi_\rho = -\mathrm{i}\hbar\frac{\delta}{\delta\rho}. \quad (12)$$

The flux $J = \frac{m_*}{2}\{\rho, u\}$ follows. Computing the flux of the vortex flow (11) we use: the $\bar\partial$-formula $\pi\delta = \bar\partial(\frac{1}{z})$ and the identity $2\sum_{i\neq j}\frac{1}{z-z_i}\frac{1}{z_i-z_j} = (\sum_i\frac{1}{z-z_i})^2 - \sum_i(\frac{1}{z-z_i})^2$. With the help of (9) a simple computation yields

$$\begin{aligned}\mathrm{P} &= \{\rho, \partial\pi_\rho\} + i\frac{q}{2\pi}\bar\partial[(\sum_i\frac{1}{z-z_i})^2 - \sum_i\frac{1}{(z-z_i)^2}] \\ &= \{\rho, \partial\pi_\rho\} + \mathrm{i}\rho\sum_j\frac{q}{z-z_j} + \mathrm{i}\frac{q}{2}\partial\rho = \mathrm{J} + \mathrm{i}\frac{q}{2}\partial\rho. \quad (13)\end{aligned}$$

We obtain the important relation between flux of the vortex flow and the flux of the fluid. Using the notation $a^*_\mu = \epsilon_{\mu\nu}a_\nu$ for 2-vectors we obtain

$$P = J + \frac{q}{4}\nabla^*\rho, \quad v = u + \frac{\Gamma}{4}\rho^{-1}\nabla^*\rho, \quad (14)$$

The shift (14) holds in the classical and the quantum cases. It has far reaching consequences [27].

The shift (14) can be seen as a similarity transformation. It preserves the volume. Hence the flow of vortices is incompressible $\nabla\cdot v = 0$ like the fluid itself.

The shift has a simple meaning. Velocity of the fluid u diverges at a core of an isolated vortex (as is in (3)). However, velocities of vortices are finite. The shift removes that singularity.

Further meaning of the shift is seen from monodromy of the wave function. Monodromy is the circulation of each particle in units of Γ. It is equal to the number of magnetic flux quantum piercing the system N_ϕ. The circulation of a particle (a vortex) around the system of remaining $N-1$ vortices is $\Gamma(N-1) = 2\pi\oint v\cdot dr$. The monodromy is $N_\phi = \beta(N-1)$, i.e., the number of zeros of the w.f. with respect to each coordinate. On the other hand the circulation $2\pi\oint u\cdot dr$ gives the total charge ΓN. The shift amounts for the difference. It simply means that a vortex does not interfere with itself. Eq. (14) can be seen as a local version of the global condition $N_\phi = \beta N - 2\bar s$, where the shift $\hbar\bar s = q/2$ [18].

We summarize the formulas for flux and velocity

$$P = \frac{1}{2}\{\rho, \nabla\pi_\rho\} - \frac{q}{2}\rho\nabla^*\varphi + \frac{q}{4}\nabla^*\rho, \quad (15)$$

$$m_*v = \nabla\pi_\rho - \frac{q}{2}\nabla^*\varphi + \frac{q}{4}\nabla^*\log\rho. \quad (16)$$

Potential φ obeys the Poisson equation $\Delta\varphi = -4\pi(\rho-\bar\rho)$. It is chosen such that flux vanishes at the ground state.

7. Chiral constituency relation. The next step is to express the flux of the vortices in terms of their density.

Using the formula $[\rho, \partial\pi_\rho] = -\mathrm{i}\hbar\partial\rho$ and the chiral relation $2\ell^2\partial^\dagger_{z_i} = -2\ell^2\overleftarrow{\partial}_{z_i} + \bar z_i$, we write

$$\{\rho, \partial\pi_\rho\} = -2\partial\pi_\rho^{\mathrm{T}}\rho + [\rho, \partial\pi_\rho] = -\mathrm{i}\hbar(\partial + \frac{\bar z}{2\ell^2})\rho.$$

Applying this formula to (15) we obtain the chiral constituency relation. We write it in two suggestive forms

$$P = -\rho\nabla^*\Psi, \quad \Psi = \frac{q}{2}[\varphi - (\frac{1}{2} - \nu)\log\rho], \tag{17}$$

$$\mathrm{P} = \frac{\mathrm{i}q}{\pi}\bar\partial\mathcal{T}, \quad \mathcal{T} = \frac{1}{2}(\partial\varphi)^2 - (\frac{1}{2} - \nu)\partial^2\varphi. \tag{18}$$

The field Ψ has a meaning of the stream function, hence vorticity is

$$\omega = -\Delta\Psi = 2\pi q[\rho - \bar\rho + \frac{1}{4\pi}(\frac{1}{2} - \nu)\Delta\log\rho]. \tag{19}$$

We observe that vorticity of the chiral flow differs from the density of vortices by the term $\propto \Delta\log\rho$. The coefficient comprises of the shift and quantum correction. The source of quantum corrections is the difference of the vorticity of the fluid $\varpi = 2\pi q[\rho - \bar\rho - \frac{\nu}{4\pi}\Delta\log\rho]$ from the density of vortices.

Integration of (17) gives the global version of the chiral condition. It connects the moment of inertia $\mathrm{L} = \int(r\times P)d^2r$ and the gyration $\mathrm{G} = \int r^2(\rho-\bar\rho)d^2r$ of the flow:

$$\ell^2\mathrm{L} = \hbar\mathrm{G} + N\ell^2\hbar(\beta - 2)$$

This form generalizes the exact sum rule of the Laughlin state at $P = 0$: $\sum_i\langle 0||z_i|^2|0\rangle = N\ell^2\,(N - (\beta-2))$.

Eq. (17) can be used to find density profiles for various coherent states [28]. Consider e.g., a quasi-hole $\psi_h = \prod_i(z_0 - z_i)\psi_0$ [1]. This state describes an anti-vortex (a hole in the sea of vortices). Its flux is $\mathrm{P} = \mathrm{i}\hbar\frac{\rho}{z - z_0}$. Eq (17) expresses its density through the 2-points function. Eq.(**??**) computes the change of the gyration by a quasi-hole: $N^{-1}\int r^2\delta\rho d^2r = -\ell^2 = -\frac{\nu}{2\pi\bar\rho}$. This result often interpreted as a fractional charge of the quasi-hole - it occupies a fraction of volume per particle. Outside of the core, the quasi-hole as a source for vorticity in a classical equation [29]

$$-\delta(r - r_0) = \beta[\rho - \bar\rho + \frac{1}{4\pi}(\frac{1}{2} - \nu)\Delta\log\rho].$$

8. Governing equation for the vortex flow. Since the density of vortices determines its flux, the continuity equation $m_*\dot\rho + \nabla\cdot P = 0$ is the only governing equation of the chiral flow. Eq.(17) provides the close form

$$\dot\rho + \frac{\Gamma}{2}\nabla\varphi\times\nabla\rho = 0.$$

This is of course the standard equation for vorticity of incompressible liquid. Peculiar effects are burred on boundaries or in initial data. A moving edge must be a stream line (17). Since stream lines are non-linearly connected to level lines of the potential φ the dynamics differs from that of a free boundary of incompressible fluid. In [5] we showed that edge dynamics is governed by Benjamin-Ono equation. Similarly, initially given vorticity is non-linearly related to the density of vortices, hence its dynamics differs from Euler flows.

9. Hamiltonian structure of vortex dynamics. Kirchhoff equations (4) are integrable. Same equations follow from different Hamiltonians [31]. One Hamiltonian comprises by kinetic energies of the fast fluid motion and slow drift of vortices $H_E = \frac{1}{2m_*}\int \mathrm{J}^\dagger\rho^{-1}\mathrm{J}d^2r$. An equivalent form of the classical version of this Hamiltonian (but without incompressible and chiral conditions), has been proposed as a description of FQHE in [9–12]. Our main result is that flows of FQH states are described by another Hamiltonian (used in [5], see also [30]), containing only slow motion of vortices [32, 33]

$$H = \frac{1}{2m_*}\int \mathrm{P}^\dagger\rho^{-1}\mathrm{P}\; d^2r. \tag{20}$$

Both yield the same Kirchhoff equations but have different energy. The difference is essential. The latter yields correct properties of the Laughlin state reflected by (14), the magneto-roton minimum in the excitation spectrum of [4] and Lorentz shear force discussed below. The former does not [34, 35].

With the help of the chiral relation (17) we obtain

$$H = \frac{1}{2m_*}\int \rho|\nabla\Psi|^2 d^2r, \tag{21}$$

where the stream function is given by (17).

The symplectic structure can be expressed in two forms: in terms of velocity or flux of the fluid, or in terms of the vortex fluid. The first structure is the canonical Heisenberg current algebra followed from (12)

$$m_*^2[u(r),\, u^\dagger(r')] = 4\pi q\delta(r - r'). \tag{22}$$

The second is an extended algebra obtained by the transformation (14) [36, 37]. It is conveniently written in terms of the flux

$$[\mathrm{P}(r), \mathrm{P}^\dagger(r')] = 2\hbar[-P\times\nabla + q\rho(2\pi\rho + \frac{1}{4}\Delta)]\delta(r - r'). \tag{23}$$

The advantage of the first structure is its canonical form. The advantage of the second structure is that operators P , $\mathrm{P}^\dagger$ are ladder operators annihilating the ground state.

The Hamiltonian (20) and the algebra (23) acting in the Bargmann space are the compact and comprehensive formulation of FQHE.

10. Structure function and excitation spectrum. We illustrate the current algebra by computing the celebrated results of [4] for the structure factor and the excitation

spectrum. Let ρ_k is a Fourier mode of a small density modulation. In the linear approximation (17) connects it to the flux mode $\mathrm{P}_k = \hbar\frac{2\mathrm{k}}{(k\ell)^2}(1-\frac{\beta-2}{4}(k\ell)^2)\rho_k$. Then project (23) onto the ground state and compute its r.h.s.. This gives the correlation of flux modes $\langle \mathrm{P}_k\mathrm{P}_k^\dagger\rangle = 2\bar\rho\hbar^2\ell^{-2}(1-\frac{\beta-2}{4}(k\ell)^2)$. Comparing we obtain the Feynman-Bijl formula [8] for the single mode approximation of the excitation spectrum $\Delta(k)\equiv\frac{\langle \mathrm{P}_k\mathrm{P}_k^\dagger\rangle}{2m_*\bar\rho^2}$, and the structure factor $s(k) = \langle\rho_k\rho_{-k}\rangle$

$$\Delta(k)=\frac{\hbar^2k^2}{2m_*s(k)},\quad s(k)=\frac{\bar\rho}{2}\frac{(k\ell)^2}{1-\frac{\beta-2}{4}(k\ell)^2}. \tag{24}$$

The excitation energy $\Delta(k) = 2\pi\beta\frac{\hbar^2}{m_*}(1-\frac{\beta-2}{4}(k\ell)^2)$ shows the magneto-roton minimum of [4], the leading order of the structure factor also agrees with [4].

11. Stress tensor and Lorentz shear force. The shift (14) causes subtle hydrodynamic phenomena. One of them is the Lorentz shear force, a conservative force acting normal to the shear flow. It enters to the conservative part of the viscous tensor, which we now compute. Initially being introduced for the integer QHE in [19] and extended to FQHE in [20, 21], the Lorentz shear force is in fact, the classical phenomena. It is a feature of vortex flow.

We cast hydrodynamics equations in the form of the conservation law with the Lorentz force $F=eE-\frac{e}{c}Bv^*$

$$\partial_t P_\mu + \nabla_\nu\Pi_{\mu\nu} = \rho F_\mu, \tag{25}$$

and compute the momentum flux $\Pi_{\mu\nu}$.

We start from the first Hamiltonian structure writing conservation law for the fluid flux J: $\partial_t J_\mu + \nabla_\nu\tilde\Pi_{\mu\nu}=0$. In this case the canonical property of the first structure (22) determines the form of the momentum flux: $\tilde\Pi_{\mu\nu} = J_\mu(m_*\rho)^{-1}J_\nu + \rho\tilde p\delta_{\mu\nu}$, while incompressibility condition determines the intrinsic pressure $\tilde p$ (all operator products are normally ordered).

We obtain the momentum flux of vortices $\Pi_{\mu\nu}$ by the similarity transformation (14). Taking time derivative of (14) and using the continuity equation, we write $\dot J_\mu - \dot P_\mu = \frac{q}{4}\nabla^*_\mu(v\cdot\nabla)\rho = \nabla_\nu(\Pi_{\mu\nu} - \tilde\Pi_{\mu\nu})$. This gives the transformation of the momentum flux $\tilde\Pi_{\mu\nu}\to\Pi_{\mu\nu} =$ $: P_\mu(m_*\rho)^{-1}P_\nu : - : \sigma_{\mu\nu}:$ and the stress tensor $\sigma_{\mu\nu}$

$$\sigma_{\mu\nu} = -\rho p\delta_{\mu\nu} - \frac{q^2}{4}\nabla_\mu\sqrt{\rho}\nabla_\nu\sqrt{\rho} + \sigma'_{\mu\nu},$$
$$\sigma'_{\mu\nu} = -\frac{q}{4}\rho(\nabla_\nu v^*_\mu + \nabla^*_\nu v_\mu).$$

The viscous part of the stress tensor, $\sigma'_{11} = -\sigma'_{22} = -\frac{q}{4}\rho(\nabla_x v_y + \nabla_y v_x)$, $\sigma'_{12} = \sigma'_{21} = \frac{q}{4}\rho(\nabla_x v_x - \nabla_y v_y)$ is the Lorentz shear force. It is traceless, hence conservative.

The effect can be interpreted in terms of semiclassical motion of electrons. A motion of electrons consists of a fast motion along small orbits and a slow motion of orbits. A shear flow strains orbits elongating them normal to the shear, boundaries and vortices. Elongation yields an addition to the Lorentz, the Lorentz shear force which acts normal to the shear and proportional to the shear.

An important consequence of the Lorentz shear force is accumulation of charges on boundaries and vortices - *overshoots.* They govern dynamics of edge modes [5].

12. Hall current. Another consequence of the Lorentz shear force is the increase of the Hall current by a non-uniform e.m. fields [22]. It follows from the conservation law (25) that in the linear approximation the Lorentz force equilibrates by the Lorentz shear force $eE - \frac{e}{c}Bv^* \approx \frac{q}{4}\Delta v^* \approx \frac{q}{4}c\Delta(E/B)$. As the result the Hall current increases. In a uniform magnetic field $j_e = \sigma_{xy}\left(1+\frac{1}{4\nu}(k\ell)^2\right)E^*_k$.

So far we neglected diamagnetic effects by counting the energy from the ground state. The ground state energy depends on the magnetic field as $-\frac{1}{2}M\hbar\omega_c\bar\rho$, where M is an orbital moment per particle. In a non-uniform density it yields the diamagnetic current $j_{dia} = -\frac{|e|\hbar}{m_e}M\nabla^*\rho$. The diamagnetic currents gives a similar contribution to the Hall conductance as the Lorentz shear force. Indeed, the density of the chiral flow is determined by the flux according to the Eq.(17). In the leading order $m_*\Delta v \approx 2\pi q\nabla^*\rho \approx m\Delta(cE^*/B)$, hence $j_{dia} \approx -\sigma_{xy}\frac{m_*}{m_e}M\ell^2\Delta E^*$. Thus diamagnetic current merely changes $\frac{1}{4\nu}\to\frac{1}{4\nu}+\frac{m_*}{m_e}M$ in the formula for the current. This result has been obtained in [22] where $m_*$was set to be equal to bare electronic mass m_e. A factor m_*/m_e offers an avenue to obtain the inertia m_* by measuring of the increase of the Hall current by a non-uniform electric or magnetic fields. Comparing it with an independently measured gap allows to check Feynman-Bijl formula (24).

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[25] Coulomb forces always emerge in flows of electronic liquids. They are essential factor in the bulk, less essential on the edge. In this paper we neglect Coulomb forces in order to unmask laws of quantum hydrodynamics.
[26] In connection to FQHE the relation between density and vorticity in a rotated Euler fluid was discussed in [12]. Recently the chiral compressible flow has been studied in [23, 24]. Gapped $\propto q\Omega$ linear waves have been observed.
[27] A similar shift of velocity has been also observed in hydrodynamics of Calogero model [15].
[28] Eq. (17) expresses flux in terms of the density and the two-points density correlation function. At the ground state where $P = 0$ the equation establishes a relation between the density and two-point function obtained in A. Zabrodin, P. Wiegmann, J.Phys. A39:8933 (2006).
[29] Incidently a similar equation exists inside the core. There the quantum corrections changes the last term to $-\frac{1}{4\pi}\nu\Delta\log\rho$. Accidently a similar equation followed from the effective action of Refs.[9, 10] erroneously featuring the term $-\frac{1}{4\pi}\nu\Delta\log\rho$ inside and outside of the vortex.
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[31] The Hamiltonian proposed by Kirchhoff himself was $\mathcal{H} = 2\Omega\int\rho\Psi d^2z = -q\Omega\sum_i\log|z_i - z_j|^2$.
[32] In formulas for the energy and momentum flux of Sec. 11 we assume the normal ordering of the density operator as in [4] : $\rho := \sum_i\sum_k e^{-i\ell^2 \mathrm{k}\partial_{z_i}} e^{-i\frac{1}{2}\bar{\mathrm{k}}z_i}$. The ordering assures that the energy vanishes in the integer case $\beta = 1$.
[33] We would like to clarify the meaning of the Hamiltonian (20). The first quantized version of the Hamiltonian is $H = \frac{1}{2m}\sum_{i=1}^N a_i^\dagger a_i$ is the second order differential operator, where $a_i = 2\hbar i(-\partial_i + \beta\sum_{j\neq i}\frac{1}{z_i - z_j}) = m\mathrm{v}_i + \partial_i\log|\psi_0|^2$ [5]. It acts in the space of symmetric holomorphic polynomials multiplied by the factor $\prod_{i>j}(z_i - z_j)^\beta$ with the inner product (8). The Hamiltonian is a complexified version of the chiral sector of Calogero model. If Q_λ is a basis in a ring of polynomials the Hamiltonian can be viewed as a semi-infinite matrix $H_{\lambda'\lambda} = (2\hbar^2/m)\sum_i\int\overline{\partial_{z_i}Q_{\lambda'}}\partial_{z_i}Q_\lambda\prod_{i>j}|z_i - z_j|^{2\beta}d\mu$. We see that the Hamiltonian can be diagonalized in terms of bi-orthogonal polynomials with the weight $\prod_{i>j}|z_i - z_j|^{2\beta}d\mu$. These polynomials are bi-orthogonal analog of Jack polynomials. Contrary to Jack polynomials, not much is known about their bi-orthogonal extension.
[34] It is instructive to analyze the energy difference of the flow and the vortex flow. Simple computations yield $H_E - H = \frac{1}{2m}\int[-\frac{q}{2}\nabla\times P + \frac{q}{2}\rho\omega + (\frac{q}{2}\nabla\sqrt{\rho})^2]d^2r$. The differential $\nabla\times P$ yields to the Lorentz shear force.
[35] Lagrangian form of the hydrodynamics is not particularly helpful since subtle quantum phenomena and the chiral constraint are hidden in the measure of integration. However, it may help to compare our approach with those of [9, 10, 23]. Let us choose the density of vortices ρ and its momentum π_ρ as canonical variables and represent the velocity as $m_* v = \nabla\pi_\rho - a + \frac{q}{4}\nabla^*\log\rho$ with a constraint $-\nabla\times a = 2\pi q(\rho - \bar{\rho})$ to match (16). The constraint enforced by the Lagrangian multiplier a_0 yields the Chern-Simons term in the Lagrangian: $L = -\rho(\dot{\pi}_\rho + a_0) + a_0\bar{\rho} - \frac{1}{2\pi q}a_0(\nabla\times a - \frac{1}{2m_*}[\rho|\nabla\pi_\rho - a|^2 - q\rho(\nabla\times a) + (\frac{q}{2}\nabla\sqrt{\rho})^2]$. Two last terms differ this Lagrangian from those of [9, 10]. They were recovered in [23]. The Lagrangian can be expressed in terms of the vertex operator $\Phi = \sqrt{\rho}e^{-\frac{i}{\hbar}\pi_\rho}$: $L = a_0\bar{\rho} - \Phi^*[(i\hbar\partial_t + a_0)\Phi - \frac{1}{2m_*}[|(i\hbar\nabla + a)\Phi|^2 + q(\nabla\times a)|\Phi|^2 + (\frac{q^2}{4} - 1)(\nabla|\Phi|)^2]$. Incompressibility condition and the chiral constraint must be added.
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[37] Elsewhere we discuss the relation of the extended algebra (23) and the Virasoro algebra of conformal field theory with the central charge $c = 1 - 6(\sqrt{\beta} - 1/\sqrt{\beta})^2$ (discussion with E. Bettelheim is acknowledged).